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REMARKS ON A PAPER OF A. ERDELYI

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REMARKS ON A PAPER OF A. ERDELYI

by

N.M. Temme

ABSTRACT

An alternative asymptotic expansion is given for an integral, which is recently considered by ERDELYI by means of fractional derivatives. The new expansion is simpler and the bounds of the remainder terms are of the same kind.

KEYWORDS & PHRASES: Asymptotic expansion, incomplete gamma function, confluent hypergeometric function.

## INTRODUCTION

1. In a recent paper [3], Professor ERDELYI considered integrals of the form

$$(1.1) \quad F(z, a) = \int_a^{\infty} e^{-z(t-a)} t^{\lambda-1} g(t) dt,$$

where  $a \geq 0$ ,  $0 < \lambda < 1$ , and  $z$  is a large parameter. In order to obtain an asymptotic expansion for  $z \rightarrow \infty$ , uniformly valid for  $a \geq 0$ , he replaced the function  $t^{\lambda-1} g(t)$  by a fractional integral  $I^{\lambda-1} f(t)$  the operator  $I^{\alpha}$  being defined by

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds.$$

By an integration by parts procedure ERDELYI obtained the uniform expansion

$$(1.2) \quad F(z, a) = Q \sum_{k=0}^{n-1} \Gamma(k+\lambda) g^{(k)}(0) z^{-k}/k! + \sum_{k=1}^{n-1} z^{-k} I^{\lambda} f^{(k)}(a) + R_n,$$

where  $Q$  is related to the incomplete gamma function

$$(1.3) \quad Q = z^{-\lambda} e^{az} \Gamma(\lambda, az) / \Gamma(\lambda).$$

The remainder  $R_n$  is estimated uniformly in  $a$  for  $a \geq 0$ . The expression  $I^{\lambda} f^{(k)}(a)$  is explicitly given in terms of derivatives of the function  $g(t)$  at  $t = 0$  and  $t = a$  as

$$(1.4) \quad I^{\lambda} f^{(k)}(a) = \sum_{m=1}^k \frac{a^{\lambda-m}}{(k-m)!} [(-1)^{m-1} \frac{\Gamma(k) \Gamma(m-\lambda)}{\Gamma(m) \Gamma(1-\lambda)} g^{(k-m)}(a) \\ - \frac{\Gamma(k+\lambda-m)}{\Gamma(\lambda-m+1)} g^{(k-m)}(0)], \quad k=1, 2, \dots$$

As remarked by ERDELYI, the expansion (1.2) could have been obtained via integration by parts of (1.1), but the explicit form (1.4) in (1.2) is not

easily obtained in that way.

In this note we give an alternative expansion of  $F(z,a)$ , which is much simpler than (1.2), and in which the bounds of the remainder terms are of the same kind. Both expansions may be derived from each other by formal rearrangement of infinite series.

2. From a numerical point of view, the expression (1.4) occurring in (1.2), is not attractive. In calculating  $I^{\lambda} f^{(k)}(a)$  high precision of the individual terms is needed to overcome numerical cancellation for small  $a$ . Furthermore, the terms  $g^{(k)}(0)$  in (1.2) and (1.4) are somewhat surprising. Of course, the singularity at  $t = 0$  due to  $t^{\lambda-1}$  gives a hint that this point may significantly contribute to the asymptotic expansion, especially when  $a$  is small. But for moderate and large values of  $a$  we cannot expect relevant information from the function values at  $t = 0$ . Moreover, the function  $g$  may not be defined for  $t < a$ .

In our opinion, the expansion (1.2) can be considerably simplified. Let us suppose that  $g \in C^n[a, a+\delta)$ ,  $|g(t)| \leq Ae^{-\alpha t}$ ,  $t > a$ , where  $\delta$ ,  $A$  and  $\alpha$  are nonnegative constants. We write

$$g(t) = \sum_{k=0}^{n-1} c_k (t-a)^k + r_n(t), \quad c_k = g^{(k)}(a)/k!.$$

Then we have

$$(2.1) \quad F(z,a) = \sum_{k=0}^{n-1} c_k F_k + R_n$$

with

$$(2.2) \quad F_k = \int_a^{\infty} e^{-z(t-a)} t^{\lambda-1} (t-a)^k dt,$$

$$(2.3) \quad R_n = \int_a^{\infty} e^{-z(t-a)} t^{\lambda-1} r_n(t) dt.$$

The first few functions  $F_k$  are easily computed. It turns out that

$$(2.4) \quad F_0 = \Gamma(\lambda)Q, \quad F_1 = (\lambda z^{-1} - a)F_0 + a^\lambda z^{-1}$$

where  $Q$  is essentially an incomplete gamma function, which is defined in (1.3). By partial integration of (2.2) we obtain

$$(2.5) \quad F_{k+1} = z^{-1}[(k+\lambda-az)F_k + akF_{k-1}], \quad k \geq 1.$$

Hence, if  $F_0$  is computed, the remaining  $F_k$  can be generated by (2.5).

The functions  $F_k$  are confluent hypergeometric functions. In the notation of [1], we have

$$(2.6) \quad \begin{aligned} F_k &= k! a^{k+\lambda} U(k+1, k+1+\lambda, az) \\ &= k! z^{-k-\lambda} U(1-\lambda, 1-\lambda-k, az). \end{aligned}$$

The second representation enables us to write for  $0 < \lambda < 1$

$$(2.7) \quad F_k = \frac{k! z^{-k-\lambda}}{\Gamma(1-\lambda)} \int_0^\infty e^{-azt} t^{-\lambda} (1+t)^{-k-1} dt,$$

from which follows, by majorizing the exponential function in the integrand by 1,

$$(2.8) \quad F_k \leq z^{-k-\lambda} \Gamma(k+\lambda).$$

As follows from (2.2), this bound is also valid for  $\lambda = 1$ .

If on  $[a, a+\delta)$  an estimate is known for  $g^{(k)}$ , say  $|g^{(k)}(t)| \leq a_k$ , and  $a, \lambda$  and  $z$  are real, then  $R_n$  in (2.3) may be majorized by  $|R_n| \leq a_n F_n$ . Using (2.8), we obtain uniformly in  $a$  for  $a \geq 0$

$$|R_n| \leq a_n z^{-n-\lambda} \Gamma(n+\lambda).$$

Consequently, in the notation of [2], we have

$$F(z, a) \sim \sum c_k F_k \quad \{z^{-k-\lambda}\} \quad \text{as } z \rightarrow \infty.$$

This shows that (2.1) is an asymptotic expansion, holding uniformly in  $a$  for  $a \geq 0$ , with respect to the asymptotic sequence  $\{z^{-n-\lambda}\}$ , which does not depend on  $a$ .

From a practical point of view, the expansion in (2.1) is more suitable than (1.2), since the coefficients  $c_k$  are simply expressed in terms of  $g^{(k)}(a)$ . Both expansions have the same bounds for the remainders. As a minor improvement, our expansion is also uniformly valid with respect to  $\lambda$  on compact sub-intervals of  $(0, 1]$ .

3. The numerical analyst may wonder if the sequence  $\{F_k\}$  can be generated in a stable way by using (2.5). The answer is affirmative, as one easily deduces from the qualitative behaviour of the linearly independent solutions of the second-order difference equation (2.5). With

$$(3.1) \quad G_k = \int_0^a e^{-zt} (t-a)^k t^{\lambda-1} dt = a^{\lambda+k} (-1)^k \frac{\Gamma(\lambda)\Gamma(k+1)}{\Gamma(k+\lambda+1)} M(\lambda, k+\lambda+1, -az),$$

the functions  $F_n, G_n$  constitute a linearly independent pair of solutions of (2.5), as follows from the asymptotic behaviour

$$(3.2) \quad F_n \sim n! z^{-n-\lambda} (1+a/n)^{\lambda+1} n^{\lambda-1}, \quad n \rightarrow \infty, \text{ uniformly in } a \geq 0,$$

and from the inequality

$$(3.3) \quad |G_n| \leq a^{n+\lambda} \Gamma(\lambda)\Gamma(n+1)/\Gamma(n+\lambda+1), \quad n=0, 1, \dots$$

Formula (3.2) is easily derived with saddle point techniques from (2.7), and (3.3) follows from (3.1) by majorizing the exponential function by 1.

The relations (3.2) and (3.3) show that, in the sense of [4], the solution  $G_n$  is a minimal solution of (2.5) and  $F_n$  a dominant solution.

4. The relation between ERDELYI's expansion (1.2) and our expansion (2.1)

can be illustrated by writing

$$F_k = P_k F_0 + Q_k a^\lambda z^{-1}, \quad k=0,1,\dots$$

$P_k$  and  $Q_k$  are polynomials in  $z^{-1}$  satisfying (2.5) with initial values  $P_0 = 1$ ,  $Q_0 = 0$ ,  $P_1 = \lambda z^{-1} - a$ ,  $Q_1 = 1$ . By using the recurrence relation it can be proved that

$$(4.1) \quad P_k = z^{-k} \sum_{j=0}^k (-az)^{k-j} \binom{k}{j} \Gamma(\lambda+j)/\Gamma(\lambda), \quad k=0,1,\dots$$

Hence, in a formal way, our expansion (2.1) can be written as

$$(4.2) \quad F(z,a) \sim F_0 \sum c_k P_k + a^\lambda z^{-1} \sum c_k Q_k.$$

With the substitution of (4.1) and using the (formal) expansion

$$g^{(j)}(t) = \sum_{k=j}^{\infty} c_k \frac{k!}{(k-j)!} (t-a)^{k-j}$$

at  $t = 0$ , we obtain, by interchanging the order of summation

$$F(z,a) \sim Q \sum z^{-k} \Gamma(k+\lambda) g^{(k)}(0)/k! + a^\lambda z^{-1} \sum c_k Q_k.$$

The first series in this expression is exactly the first series of ERDELYI in (1.2). The second series is much more complicate, but it can surely be identified with the corresponding series of ERDELYI.

The conclusion is, that the series (1.2) and (2.1) may be derived from each other by formal rearrangement of infinite series.

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